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Exact Mean-Field Solutions of the Asymmetric Random Average Process

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We consider the asymmetric random average process (ARAP) with continuous mass variables and parallel discrete time dynamics studied recently by Krug/Garcia and Rajesh/Majumdar [both *J. Statist. Phys.* **99** (2000)]. The model is defined by an arbitrary state-independent fraction density function $\phi(r)$ with support on the unit interval. We examine the exactness of mean-field steady-state mass distributions in dependence of ϕ and identify as a conjecture based on high order calculations the class $\mathscr{M}$ of density functions yielding product measure solutions. Additionally the exact form of the associated mass distributions $P(m)$ is derived. Using these results we show examplary the exactness of the mean-field ansatz for monomial fraction densities $\phi(r) = (n-1)\, r^{n-2}$ with $n \geqslant 2$. For verification we calculate the mass distributions $P(m)$ explicitly and prove directly that product measure holds. Furthermore we show that even in cases where the steady state is not given by a product measure very accurate approximants can be found in the class $\mathscr{M}$.

KEY WORDS: Non-equilibrium physics; stochastic systems; interacting particle system; random average process; q model; invariant product measure; discrete time dynamics; exact solution.

1. INTRODUCTION

Interacting particle systems far from equilibrium represent due to their wide applications in physics and other related topics like traffic flow modeling[(1)] or force propagation in granular media[(5)] a popular theoretical research field.[(8)] In this paper we focus on a model studied recently by Krug and Garcia[(4)] and Rajesh and Majumdar[(7)] that is closely related to the

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q model introduced by Coppersmith *et al.*[3] It describes the movement of particles on the real line according to a given probability distribution depending on the distance of two particles. This system is equivalent to a stick model where the height of the sticks represents the particle gap.[4, 7] Below we present a brief description for completeness and demarcation of our framework.

The asymmetric random average process (ARAP) is defined on a one-dimensional periodic lattice with L sites. Each site i carries a non-negative continuous mass variable m_i. In every time step $t \to t+1$ for each site a random number r_i is generated from a time- and site-independent probability distribution ϕ defined on $[0, 1]$. The fraction r_i determines the amount of mass $r_i m_i$ transported from site i to site $i+1$ (asymmetric shift).

In the following we concentrate on the infinite system in the thermodynamic limit, i.e., $L \to \infty$ and $M = \sum_i m_i \to \infty$ with finite constant density $\rho = \frac{M}{L}$. Furthermore we examine only time independent steady state dynamics.

In case of a uniform fraction density $\phi(r) = 1$ mass distributions and moments have been calculated in mean-field approximation for different types of dynamics.[4, 6, 7] Although these analytical results show excellent agreement with numerical simulations, one can prove that for random sequential update a product measure ansatz fails.[7] For the fully parallel update, however, one can adopt the result of ref. 3 where the exactness of the mean-field approach is proven in the context of force fluctuations in bead packs described by the q model.[3, 2]

In this paper we like to reinforce investigations by considering arbitrary state independent fraction density functions $\phi = \phi(r)$ and focussing on the fully parallel update only. Such density functions are relevant for practical applications, e.g., as approximation for the ARAP with a finite mass cut-off[11] or for describing suitable bead pack problems.

In Section 2 a functional equation acting on the Laplace-space of the single-site mass distributions $P(m)$ is derived. Product measure holds if a mean-field ansatz is a solution of this condition.

In Section 3 we determine the set $\mathscr{M}$ of all fraction probability densities $\phi(r)$ yielding product measure states. This result represents a conjecture based on exact high order calculations. Now statements about the exactness of mean-field are possible without solving the master equation. We also derive the corresponding mass distributions $P(m)$.

As an example the ARAP with monomial ϕ-function is studied (Section 4). We calculate $P(m)$ and prove the exactness of mean-field by using the explicit form of $P(m)$ on the one hand and the criterion of the last section on the other hand. Due to the fact that the n dimensional q model with uniform distributed q's leads to the same mass distribution[3] its relationship with the ARAP is discussed, too.

In Section 5 we demonstrate that the results obtained in the previous sections can be used to obtain very accurate approximations for the mass distributions of ARAPs which do not lead to product measures.

In Section 6 we conclude with a summary and give a brief outlook for further work.

2. A FUNCTIONAL EQUATION OF EXACT MEAN-FIELD SOLUTIONS

In this section we derive a functional equation (see Eq. (7) below) for determining and testing mean-field solutions.

The fundamental element of all upcoming considerations is the master equation. For the single site mass distribution $P(m)$ in the stationary state it reads[7]

$$P(m_2') = \int_0^\infty dm_1 \int_0^\infty dm_2\, P(m_1, m_2) \int_0^1 dr_1 \int_0^1 dr_2\, \phi(r_1)\, \phi(r_2) \times \delta(m_2' - [r_1 m_1 + (1-r_2)\, m_2]). \tag{1}$$

The δ-function ensures mass conservation. We have assumed translational invariance so that the distribution is site-independent.

By the mean-field ansatz $P(m_1, m_2,...) = \prod_i P(m_i)$ Eq. (1) determines the single site distribution $P(m)$. The resulting product measure is exact if the mean-field ansatz holds for all joint probabilities in the stationary state, too, i.e., $P(m)$ has to satisfy

$$\prod_{i=2}^{k} P(m_i') = \left\{ \prod_{i=1}^{k} \int_0^\infty dm_i\, P(m_i) \int_0^1 dr_i\, \phi(r_i) \right\} \times \prod_{i=2}^{k} \delta(m_i' - [r_{i-1} m_{i-1} + (1-r_i)\, m_i]) \tag{2}$$

for all $k \in \mathbb{N}_{\geqslant 2}$. For $k = 2$ this equation reduces to (1). This appears to be an infinite set of conditions, but Laplace transforming $P(m_1, m_2, ...)$ reduces (2) to just one functional equation. By introducing the k-dimensional Laplace-transform

$$Q(s_1,..., s_k) \equiv \int_0^\infty d^k m\, P(m)\, e^{-(m,s)}, \tag{3}$$

where $(m, s) = \sum_{i=1}^{k} m_i s_i$, and using the map

$$F_Q(s, \tilde{s}) \equiv \int_0^1 dr\, \phi(r)\, Q((1-r)\, s + r\tilde{s}) \tag{4}$$

Eq. (2) reads in Laplace space

$$\prod_{i=1}^{k} Q(s_i) = F_Q(0, s_1) \cdot \prod_{i=1}^{k-1} F_Q(s_i, s_{i+1}) \cdot F_Q(s_k, 0) \tag{5}$$

for $k \in \mathbb{N}$.

By a straightforward proof we now show that the conditions for $k \neq 2$ are redundant. The $k = 1$-equation

$$Q(s) = F_Q(s, 0) \cdot F_Q(0, s) \tag{6}$$

is used to determine Q. We rewrite the $k = 2$-criterion using (6) and obtain

$$F_Q(s_1, s_2) = F_Q(s_1, 0) \cdot F_Q(0, s_2). \tag{7}$$

Applying (6) and (7) proves the validity of (5) for all $k \geqslant 3$ and using the identity $Q(s) = F_Q(s, s)$ Eq. (6) becomes a special case of (7). So (7) is the only necessary equation to determine a mean-field solution ($s_1 = s_2$) and check its accuracy ($s_1 \neq s_2$).

3. EXACT MEAN-FIELD SOLUTIONS

In this section we derive the set of density functions $\phi(r)$ that result in product measure steady states. This yields a more useful criterion for determining the exactness of a mean-field solution without calculating and verifying $Q(s)$ by condition (7). In addition we derive the mass distribution $P(m)$.

We start by proving that Eq. (6) has always a unique solution in the space of functions that are analytical in the origin. Intuitively one supposes this feature because the mass moments

$$m_n \equiv \langle m^n \rangle_{P(m)} = \int_0^\infty dm\, m^n P(m) \tag{8}$$

are (formally) generated by $m_n = (-1)^n Q^{(n)}(0)$. But a priori we do not know if all derivatives $Q^{(n)}$ of the moment function $Q(s)$ exist or ensure convergence. The moments $m_0 = 1$ and $m_1 = \rho$ are determined by the normalization and the density ρ, respectively.

We first represent the moment function as a (formal) power series, i.e., $Q(s) = \sum_n a_n s^n$. The coefficients a_n are related to the moments (8) by $a_n = (-1)^n \frac{m_n}{n!}$ and thus we have $a_0 = 1$ and $a_1 = -\rho$. The remaining coefficients are determined with help of a recurrence relation obtained by inserting the series representation into (6):

$$a_n = \frac{1}{1-\mu_{n,0}-\mu_{0,n}} \sum_{k=1}^{n-1} \mu_{k,0}\mu_{0,n-k} a_k a_{n-k} \qquad (\forall n \geqslant 2). \tag{9}$$

Here $\mu_{n,m}$ are generalized moments of the fraction density ϕ defined by

$$\mu_{n,m} \equiv \langle r^n(1-r)^m \rangle_{\phi(r)} = \int_0^1 dr\, \phi(r)\, r^n (1-r)^m. \tag{10}$$

We assume $\mu_{n,m} > 0$ in the following which is equivalent to $\phi(r) \neq 0$ for $r \in (0,1)$. $\mu_{n,m} = 0$ does not occur for continuous distributions, but e.g., for $\phi(r) = p\delta(r) + (1-p)\,\delta(1-r)$. This ARAP is trivial for $p = 0$ or $p = 1$ and not solvable under mean-field assumptions for $0 < p < 1$. From

$$1 - \mu_{n,0} - \mu_{0,n} \overset{(10)}{=} \sum_{k=1}^{n-1} \binom{n}{k} \mu_{k,n-k} > 0 \qquad (\forall n \geqslant 2) \tag{11}$$

we conclude that all a_n are well-defined and the solution of (6) is unique. By the formula of Cauchy–Hadamard we then show that Q is holomorphic in $s = 0$: We start by proving the lemma

$$|a_n| \leqslant D^{n-1} c_n \rho^n \qquad (\forall n \geqslant 1) \tag{12}$$

with $D \equiv \frac{1}{1-\mu_{2,0}-\mu_{0,2}}$ and the density ρ. Here $\{c_n\}_{n\in\mathbb{N}}$ are the Catalan numbers[10] fulfilling the equations

$$c_1 = 1 \qquad \text{and} \qquad c_n = \sum_{k=1}^{n-1} c_k c_{n-k} = \frac{1}{n}\binom{2(n-1)}{n-1} \qquad (\forall n \geqslant 2). \tag{13}$$

Inserting (12) into (9) using (13) and the fact that $\mu_{n,m} > \mu_{n+k_1,m+k_2}$ for all $k_i \in \mathbb{N}$ shows inductively the validity of the lemma (12). From $\lim_{n\to\infty} \sqrt[n]{c_n} = 4$ we conclude that the series expansion $Q(s) = \sum_n a_n s^n$ has a positive radius of convergence.

After proving that mean-field solutions are always representable as power series we now try to express the exact solutions in terms of the density function ϕ. Inserting $Q(s) = \sum_n a_n s^n$ into (7) yields an infinite set of conditions

$$\binom{n}{k} \mu_{k,n-k} a_n = \mu_{k,0}\mu_{0,n-k} a_k a_{n-k} \qquad (\forall n \in \mathbb{N}_0,\ \forall k \in \{0,\ldots,n\}). \tag{14}$$

In the following we refer to a special equation of (14) as (n, k)-condition or equation of order n. Summing over $k=1,\ldots, n-1$ in (14) yields condition (9). This shows again that (7) includes formula (6) and implies convergence of $Q(s)$.

Now we are confronted with the problem that a_n is determined by $n+1$ equations. Since the $(n, 0)$- and (n, n)-conditions match identities there are effectively $n-1$ equations to be fulfilled for $n \geqslant 2$. Thus for $n \geqslant 3$ the occurence of inconsistencies is possible and for arbitrary ϕ or moments $\mu_{n,m}$ we see by explicit calculation contradictions in order $n=3$ already. Is it possible to find a set of moments $\{\mu_{n,m}\}$ such that (14) is satisfied for all (n, k)?

Assuming that all (n, k)-conditions yield the same a_n we see from (14) that the function

$$f(n,k) \equiv \frac{\mu_{k,0}\mu_{0,n-k}a_k a_{n-k}}{\binom{n}{k}\mu_{k,n-k}}. \tag{15}$$

is independent of k. Therefore the $n-2$ consistency equations

$$f(n,1) = f(n,2) = \cdots = f(n,n-1) \equiv a_n \tag{16}$$

have to be satisfied. The definition (10) implies that $\mu_{n,m}$ can be expressed by $\mu_{j,0}$ only:

$$\mu_{n,m} = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \mu_{n+j} \qquad \text{with} \quad \mu_j \equiv \mu_{j,0}. \tag{17}$$

Together with (14) it follows inductively that $a_n = a_n(\mu_0,\ldots, \mu_n)$. This leads to a successive constructive approach in n: We try to find a μ_n (depending on $\mu_0,\ldots, \mu_{n-1}$) that solves all conditions (16) of order n starting with $n=3$ and repeat this for all upcoming orders $n=4, 5,\ldots$.

The $n-2$ equations (16) are linear in μ_n. This ensures uniqueness of a possible solution. For arbitrary $k, \tilde{k}=1,\ldots, n-1$ with $k \neq \tilde{k}$ we obtain

$$\mu_n = \frac{\binom{n}{k} h_{n,k} g_{n,\tilde{k}} - \binom{n}{\tilde{k}} h_{n,\tilde{k}} g_{n,k}}{\binom{n}{\tilde{k}} (-1)^{n-\tilde{k}} g_{n,k} - \binom{n}{k} (-1)^{n-k} g_{n,\tilde{k}}}. \tag{18}$$

Here $g_{n,k} \equiv \mu_{k,0}\mu_{0,n-k}a_k a_{n-k}$ and $h_{n,k} \equiv \sum_{j=0}^{n-k-1} \binom{n-k}{j}(-1)^j \mu_{k+j}$ only depend on $\mu_1,\ldots, \mu_{n-1}$. Thus (18) gives us the desired recursion relation to determine all moments μ_n. However, we have to check whether the r.h.s. of (18) is independent of k and $\tilde{k}$. We have done this up to order $n=10$ by computer algebra and conjecture the validity for all n. Our results furthermore lead us to conjecture the following form of the solution of the density moments:

$$\mu_n = \prod_{l=0}^{n-1} \frac{l+\lambda_1}{l+\lambda_2} = \frac{\Gamma(n+\lambda_1)}{\Gamma(\lambda_1)} \frac{\Gamma(\lambda_2)}{\Gamma(n+\lambda_2)} \tag{19}$$

with

$$\lambda_1 = \mu_1 \frac{\mu_1-\mu_2}{\mu_2-\mu_1^2}, \qquad \lambda_2 = \frac{\mu_1-\mu_2}{\mu_2-\mu_1^2}, \tag{20}$$

which again has been checked up to $n=10$. Note that μ_1 and μ_2 are free parameters that only have to be choosen with respect to the general moment properties

$$1 > \mu_1 > \mu_2 \geqslant \mu_1^2. \tag{21}$$

The special case $\mu_1 = \mu_2^2$ yields $\mu_n = \mu_1^n$ representing $\phi(r) = \delta(r-\mu_1)$ which leads to $Q(s) = \exp(-\rho s)$ and thus to the mass distribution $P(m) = \delta(m-\rho)$. So we assume $\mu_2 > \mu_1^2$ in the following. Under these restrictions $\mu_{n\geqslant 3}$ is pole-free and satisfies $0 < \mu_{n+1} < \mu_n$ as demanded.

Equations (19)–(21) define the set $\mathscr{M}$ of all fraction densities $\phi(r)$ yielding product measure steady-state distributions. So Eqs. (19) and (20) represent a powerful criterion for determining the exactness of a mean-field ansatz: We only have to calculate the moments μ_n of the fraction density (if they are not already given) and check consistency with (19) and (20)—without even calculating the mean-field mass distribution or its Laplace-transform!

Note that the parametrization of $\mathscr{M}$ in terms of μ_1 and μ_2 is arbitrary and a consequence of our construction—other parametrizations are possible. For symmetric densities $\phi(r) = \phi(1-r)$ the space $\mathscr{M}$ reduces to one dimension because $\mu_1 = \frac{1}{2}$ is fixed.

After determining the class $\mathscr{M}$ of fraction densities leading to a product measure we now like to calculate the single site mass distribution $P(m)$ for these $\phi \in \mathscr{M}$. From the $(n+1, n)$-condition (14) we derive the recurrence relation

$$\lambda_2(n+1)\, a_{n+1} + \rho(n+\lambda_2)\, a_n = 0 \tag{22}$$

that is valid for all $n \in \mathbb{N}_0$. Equation (22) corresponds to a first order differential equation for the moment function $Q(s) = \sum_n a_n s^n$:

$$(\lambda_2 + \rho s)\, Q'(s) + \lambda_2 \rho Q(s) = 0. \tag{23}$$

Using the boundary condition $Q(0) = 1$ we obtain

$$Q(s) = \frac{1}{\left(1 + \frac{\rho}{\lambda_2}\, s\right)^{\lambda_2}} \tag{24}$$

or, by calculating the inverse Laplace-transform,

$$P(m) = \frac{\lambda_2^{\lambda_2}}{\Gamma(\lambda_2)} \frac{1}{\rho} \left(\frac{m}{\rho}\right)^{\lambda_2 - 1} e^{-\lambda_2 \frac{m}{\rho}}. \tag{25}$$

In contrast to μ_n depending on both λ_1 and λ_2—see Eq. (19)—$P(m)$ is a function of λ_2 only. So ARAPs with λ_2 fixed and λ_1 arbitrary have identical mass distributions (25).

4. EXPLICIT SOLUTION FOR MONOMIAL DENSITY FUNCTION $\phi(r)$

In this section we derive the solution of the ARAP with density function

$$\phi_n(r) = (n-1)\, r^{n-2} \qquad (\forall n \in \mathbb{N}_{\geqslant 2}) \tag{26}$$

in a closed form ($n-1$ is the normalization constant) and prove the exactness of the product measure $\prod_k P(m_k)$ both explicitly and using the criterion (19)–(20). Additionally a brief comparison between ARAP and the q model is presented.

We start by constructing the analytic solution of the functional equation (6) explicitly. For $n = 2$, where ϕ_n reduces to a constant distribution, we refer to refs. 3, 4, and 7 and find

$$Q_2(s) = \frac{1}{\left(1 + \frac{\rho}{2}\, s\right)^{2}}. \tag{27}$$

To solve the $n = 3$-problem we generalize the method used in refs. 3, 4, and 7. Defining the functions

$$V(s) = \int_0^1 dr\, Q_3(rs) \qquad \text{and} \qquad W(s) = \int_0^1 dr\, rQ_3(rs) \tag{28}$$

the functional equation (6) transforms into

$$Q_3(s) = 4W(s)(V(s) - W(s)). \tag{29}$$

From (28) we derive

$$sV'(s) + V(s) = Q_3(s) = sW'(s) + 2W(s) \tag{30}$$

which implies the following relation between V and W:

$$V(s) = W(s) + \frac{1}{s}\int W(s)\,ds. \tag{31}$$

Defining the antiderivative $f(s) \equiv \int W(s)\,ds$ and inserting (31) and (30) into (29) yields the nonlinear differential equation

$$s^2 f''(s) + 2sf'(s) - 4f'(s)\,f(s) = 0 \tag{32}$$

with boundary conditions $f'(0) = \frac{1}{2}$ and $f''(0) = -\frac{1}{3}\rho$. This results in $f(s) = \frac{s}{2}(1 + \frac{\rho}{3}s)^{-1}$, e.g., solved by power series ansatz, and we get

$$Q_3(s) = \frac{1}{\left(1 + \frac{\rho}{3}s\right)^3}. \tag{33}$$

The results (27) and (33) suggest the assumption

$$Q_n(s) = \frac{1}{\left(1 + \frac{\rho}{n}s\right)^n}, \tag{34}$$

for general n. Q_n fulfills the initial conditions $Q_n(0) = 1$ and $Q_n'(0) = -\rho$. By a straightforward induction in n (using partial integration) we are able to prove

$$F_{Q_n}(s, 0) = \frac{1}{1 + \frac{\rho}{n}s} \qquad \text{and} \qquad F_{Q_n}(0, s) = \frac{1}{\left(1 + \frac{\rho}{n}s\right)^{n-1}} \tag{35}$$

and see with (34) that (6) is valid.

The next step is to verify the functional equation (7). This is done again straightforwardly by induction in n. So (34) represents the exact solution of the ARAP with fraction density (26).

Now we rederive (34) using the mean-field criterion (19)–(20). Calculating the moments of ϕ_n exactly is an easy task and yields

$$\mu_k = \frac{n-1}{n-1+k}. \tag{36}$$

Especially we have $\mu_1 = \frac{n-1}{n}$ and $\mu_2 = \frac{n-1}{n+1}$ and, using (20), we obtain $\lambda_1 = n-1$ and $\lambda_2 = n$.

The exact form (36) of the moments is reproduced by taking into account (19). Thus the monomial density functions ϕ_n are elements of $\mathscr{M}$. Using $\lambda_2 = n$ shows the equivalence of (24) and (34) after all.

Additionally this example confirms the validity of the conjecture (19)–(20).

For completeness we also give the explicit form of the single-site mass distribution for the monomial density functions (26):

$$P_n(m) = \frac{n^n}{(n-1)!} \frac{m^{n-1}}{\rho^n} e^{-n\frac{m}{\rho}}. \tag{37}$$

Finally we like to mention that Q_n also satisfies the relation

$$Q_n(s) = (F_{Q_n}(0, s))^n = \left[\int_0^1 dr\, \phi(r)\, Q_n(rs)\right]^n. \tag{38}$$

This functional equation represents the master equation of the n ancestor q model with uniform distributed q's and was explicitly solved in ref. 3. Because of the formal difference of the underlying equations (6) and (38) the coincidence of the corresponding solution (34) is remarkable. Otherwise this result is pointing to a deeper relationship between n dimensional q model and ARAP.

As already noticed in ref. 4 the two ancestors q model ($n = 2$) corresponds to an ARAP with symmetric density function $\phi(r) = \phi(1-r)$. Coppersmith *et al.*[3] identified a set of mean-field solutions generated by monomial distributions, i.e., $f(q_{ij}) = q^m$, yielding $\phi(r) = \frac{(2m+1)!}{(m!)^2} r^m(1-r)^m$ and it is easy to see that these densities are also elements of $\mathscr{M}$.

5. APPROXIMATIVE MASS DISTRIBUTIONS FOR ARBITRARY DENSITY FUNCTIONS

As an application of our results, we construct approximative mass distributions for arbitrary density functions ϕ. This is done by calculating the parameter $\lambda_2 = \lambda_2(\phi)$ with the help of (20) (using the exact moments

μ_1 and μ_2 of ϕ) and taking the corresponding mean-field solution (25) as an approximation. To illustrate this method and estimate its quality we consider two examples.

Our discussion starts with density functions being convex combinations of elements of $\mathscr{M}$. We restrict here to the special case

$$\phi_c = (1-c)\,\phi_2 + c\phi_3, \qquad c \in [0, 1] \tag{39}$$

with the monomial density functions ϕ_n defined in (26). This convex combination of probability densities conserves their basic properties like normalization or positivity. Calculating the first and second moment of ϕ_c yields $\lambda_2 = \frac{6}{3-c^2}$. Inserting this result into (25) generates c-dependent approximations P_c.

Comparing the distribution P_c with numerical data shows an excellent agreement between approximation and the results of Monte Carlo simulations for all values of c (see Fig. 1). Only for small masses m systematical differences occur.

Furthermore one can prove that $\phi_c \notin \mathscr{M}$ for all $0 < c < 1$. This is most easily seen by comparing the third moment μ_3 of (39) with the corresponding mean-field expression (19). So the excellent agreement of the data match is far from trivial. Nevertheless ϕ_c is an interpolation between exact mean-field solutions and may inherit some of their properties.

Our second example represents the simplest version of an ARAP with cut-off. It is based on the model with uniform fraction density, but enhanced by

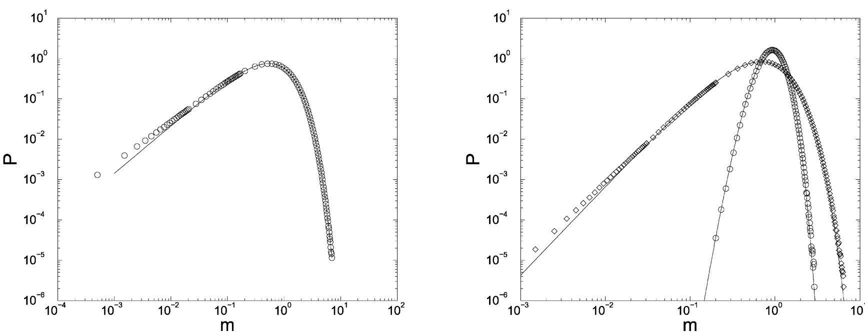

Fig. 1. Analytical (—) and numerical (◇, ○) mass distributions $P(m)$ of the convex combined ARAP ($c = 0.5$, left diagram) and the truncated ARAP ($r_0 = 0.5$ (◇) and $r_0 = 0.1$ (○), right diagram). The analytical curves are appropriate approximants taken from the mean-field class $\mathscr{M}$. The numerical results are obtained by Monte-Carlo simulations of systems with size $L = 1000$ and random initial condition. After 10^4 steps the distribution was measured for 10^7 timesteps. A log-log plot is used to exhibit the deviations for small masses m which can hardly be seen in a conventional representation.

an additional parameter $r_0 \in [0, 1]$. The cut-off r_0 controls the movement in the following way: A stick fragment $r_i m_i$ is only transferred if $r_i \leqslant r_0$. The corresponding density function takes the form:

$$\phi_{r_0}(r) = (1 - r_0)\,\delta(r) + H(r_0 - r) \tag{40}$$

where $H(r)$ is the Heaviside step-function. A detailed discussion of several truncated ARAPs can be found in ref. 11.

If $r_0 = 1$ we obtain the free ARAP which is exactly solvable by product measure ansatz.[3] For $0 < r_0 < 1$ one can show that $\phi_{r_0} \notin \mathscr{M}$. In the case $r_0 = 0$, where no motion is possible, the mean-field condition (7) reduces to an identity—so any distribution represents an exact mean-field solution. In this sense (40) interpolates between ARAPs with product measure steady state as in the first example, but the construction is not a convex combination.

The approximants P_{r_0} are calculated as described above and match the Monte-Carlo simulation data perfectly again except deviations for small m (see Fig. 1).

6. CONCLUSIONS

We have studied the asymmetric random average process with arbitrary fraction density ϕ. Based on the analysis of the functional equation (7) we have identified all ARAPs where the stationary mass distribution $P(m)$ is given by a product measure. The corresponding ϕ-functions form the mean-field class $\mathscr{M}$. They are given by their moments $\mu_n = \langle r^n \rangle_\phi$ which depend on two free parameters, λ_1 and λ_2, related to the first moments μ_1 and μ_2 (20). These can be chosen arbitrarily subject only to the general conditions (21). For the product measure ARAPs also the stationary mass distribution $P(m)$ can be calculated explicitly (25). Surprisingly it depends only on the parameter λ_2.

The presented approach is rigorous except for a formal proof of (19) which we have conjectured on the basis of computer algebra calculations.

As shown our results can be used to obtain accurate approximations for ARAPs which do not lead to exact product measures: instead of calculating mean-field approximations by (6) one can use the corresponding distribution (25) with appropriately determined $\lambda_2 = \lambda_2(\phi)$ as a first guess. A detailed discussion of $\mathscr{M}$ could explain why this method often matches perfectly with numerical simulation data. We suppose that arbitrary choosen ϕ lie "near" to the subspace $\mathscr{M}$ with respect to a suitable chosen norm.

We like to mention that our approach can easily be adopted to the ARAP with random sequential update. Since a mean-field ansatz breaks down even for the simple model with uniform ϕ-function[7] we have not done calculations for this case. Nevertheless the existence of product measure ARAPs with continuous time dynamics is also possible. Furthermore one could try to transfer our calculations to the q model to supplement the findings of ref. 3. It would be interesting to use our results for the calculation of other properties, too, e.g., the tracer diffusion coefficient.[9]

The calculation of exact solutions of ARAPs with arbitrary $\phi(r)$ seems to be the final step of work. Vanishing of the 2-site-correlation functions[4] and the excellent quality of mean-field approximations have already pointed to a distribution that deviates only slightly from product measure form. Supplementary our work gives by Eq. (25) a rough shape of the general solutions.

Finally it would be interesting to extend our considerations to state dependent density functions, i.e., $\phi = \phi(r, \{m_i\})$, and especially to local probabilities $\phi(r, m)$. These models are for example interesting for truncated asymmetric random average processes.[11]

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